Supplementary Material for Tensor Factorization for Low-Rank Tensor Completion

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I. Proof of Lemma 2

Proof. Assume that $\text{rank}_m(\mathcal{F}) = \boldsymbol{k}^{\mathcal{F}}$, where $\boldsymbol{k}_i^{\mathcal{F}} = \text{rank}(\bar{\boldsymbol{F}}^{(i)})$ $(i = 1, \cdots, n_3)$. Then, we have $\hat{k} = \max(\boldsymbol{k}_1^{\mathcal{F}}, \cdots, \boldsymbol{k}_{n_3}^{\mathcal{F}})$. Since the rank of the matrix $\bar{\boldsymbol{F}}^{(i)}$ is $\boldsymbol{k}_i^{\mathcal{F}}$, it can be factorized into the matrix product form $\bar{\boldsymbol{F}}^{(i)} = \hat{\boldsymbol{G}}^{(i)}\hat{\boldsymbol{H}}^{(i)}$, where $\hat{\boldsymbol{G}}^{(i)} \in \mathbb{C}^{n_1 \times \boldsymbol{k}_i^{\mathcal{F}}}$ and $\hat{\boldsymbol{H}}^{(i)} \in \mathbb{C}^{\boldsymbol{k}_i^{\mathcal{F}} \times n_2}$ are the i-th block diagonal matrices of $\hat{\boldsymbol{G}} \in \mathbb{C}^{n_1 n_3 \times (\sum_{i=1}^{n_3} \boldsymbol{k}_i^{\mathcal{F}})}$ and $\hat{\boldsymbol{H}} \in \mathbb{C}^{(\sum_{i=1}^{n_3} \boldsymbol{k}_i^{\mathcal{F}}) \times n_2 n_3}$, respectively, and they meet $\text{rank}(\hat{\boldsymbol{G}}^{(i)}) = \text{rank}(\hat{\boldsymbol{H}}^{(i)}) = \boldsymbol{k}_i^{\mathcal{F}}$. Then, let $\bar{\boldsymbol{G}}^{(i)} = [\hat{\boldsymbol{G}}^{(i)}, \boldsymbol{0}] \in \mathbb{C}^{n_1 \times \hat{k}}$ and $\bar{\boldsymbol{H}}^{(i)} = [\hat{\boldsymbol{H}}^{(i)}; \boldsymbol{0}] \in \mathbb{C}^{\hat{k} \times n_2}$, where $\bar{\boldsymbol{G}}^{(i)} \in \mathbb{C}^{n_1 \times \hat{k}}$ and $\hat{\boldsymbol{H}}^{(i)} \in \mathbb{C}^{\hat{k} \times n_2}$ are the i-th block diagonal matrices of $\bar{\boldsymbol{G}} \in \mathbb{C}^{n_1 n_3 \times \hat{k} n_3}$ and $\bar{\boldsymbol{H}} \in \mathbb{C}^{\hat{k} n_3 \times n_2 n_3}$, respectively. Therefore, we have $\bar{\boldsymbol{C}} = \hat{\boldsymbol{G}}\hat{\boldsymbol{H}} = \bar{\boldsymbol{G}}\bar{\boldsymbol{H}}$. From Lemma 1, we know that for any three tensors of proper sizes, $\bar{\boldsymbol{E}} = \bar{\boldsymbol{X}}\bar{\boldsymbol{Y}}$ and $\mathcal{E} = \mathcal{X} * \mathcal{Y}$ are equivalent. Therefore, we can obtain $\mathcal{C} = \mathcal{G} * \mathcal{H}$, where $\mathcal{G} \in \mathbb{R}^{n_1 \times \hat{k} \times n_3}$ and $\mathcal{H} \in \mathbb{R}^{\hat{k} \times n_2 \times n_3}$ are two tensors of smaller sizes and they meet $\text{rank}_{\text{t}}(\mathcal{G}) = \text{rank}_{\text{t}}(\mathcal{H}) = \hat{k}$.

Now we prove the second property. Assume that $\text{rank}_m(\mathcal{A}) = \boldsymbol{r}^{\mathcal{A}}$ and $\text{rank}_t(\mathcal{A}) = \hat{r}^{\mathcal{A}}$, where $\boldsymbol{r}_i^{\mathcal{A}} = \text{rank}(\bar{\boldsymbol{A}}^{(i)})$ $(i = 1, \cdots, n_3)$ and $\hat{r}^{\mathcal{A}} = \max(\boldsymbol{r}_1^{\mathcal{A}}, \cdots, \boldsymbol{r}_{n_3}^{\mathcal{A}})$. Let $\mathcal{Z} = \mathcal{A} * \mathcal{B}$. Similarly, suppose that $\text{rank}_m(\mathcal{B}) = \boldsymbol{r}^{\mathcal{B}}$, $\text{rank}_t(\mathcal{B}) = \hat{r}^{\mathcal{B}}$, $\text{rank}_m(\mathcal{Z}) = \boldsymbol{r}^{\mathcal{Z}}$, and $\text{rank}_t(\mathcal{Z}) = \hat{r}^{\mathcal{Z}}$. On the other hand, if $\boldsymbol{M} \in \mathbb{C}^{n_5 \times n_6}$ and $\boldsymbol{N} \in \mathbb{C}^{n_6 \times n_7}$ are two matrices, then we have $\text{rank}(\boldsymbol{M}\boldsymbol{N}) \leq \min(\text{rank}(\boldsymbol{M}), \text{rank}(\boldsymbol{N}))$. Thus, we have $\boldsymbol{r}_i^{\mathcal{Z}} = \text{rank}(\bar{\boldsymbol{Z}}^{(i)}) = \text{rank}(\bar{\boldsymbol{A}}^{(i)}\bar{\boldsymbol{B}}^{(i)}) \leq \min(\text{rank}(\bar{\boldsymbol{A}}^{(i)}), \text{rank}(\bar{\boldsymbol{B}}^{(i)})) = \min(\boldsymbol{r}_i^{\mathcal{A}}, \boldsymbol{r}_i^{\mathcal{B}})$. We can further obtain that $\hat{r}^{\mathcal{Z}} = \max(r_1^{\mathcal{Z}}, \cdots, r_{n_3}^{\mathcal{Z}}) \leq \min(\hat{r}^{\mathcal{A}}, \hat{r}^{\mathcal{B}})$. So the inequality $\text{rank}_t(\mathcal{A} * \mathcal{B}) \leq \min(\text{rank}_t(\mathcal{A}), \text{rank}_t(\mathcal{B}))$ in Lemma 2 holds. ☐

II. Proof of Theorem 2

Before we prove Theorem 2, we first present two lemmas. Since $\hat{\boldsymbol{X}}^{(i)}$ and $\hat{\boldsymbol{Y}}^{(i)}$ are the i-th block diagonal matrices of $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{Y}}$, respectively, for brevity, we rewrite the Eq. (6) and (7) as $\hat{\boldsymbol{X}}^{k+1} = \bar{\boldsymbol{C}}^k(\hat{\boldsymbol{Y}}^k)^* \left(\hat{\boldsymbol{Y}}^k(\hat{\boldsymbol{Y}}^k)^*\right)^{\dagger}$ and $\hat{\boldsymbol{Y}}^{k+1} = \left((\hat{\boldsymbol{X}}^{k+1})^*\hat{\boldsymbol{X}}^{k+1}\right)^{\dagger}(\hat{\boldsymbol{X}}^{k+1})^*\bar{\boldsymbol{C}}^k$, respectively.

Lemma 3. *Assume that the sequence $\{(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \mathcal{C}^k)\}$ is generated by Algorithm 1, i.e., they meet $\hat{\boldsymbol{X}}^{k+1} = \bar{\boldsymbol{C}}^k(\hat{\boldsymbol{Y}}^k)^* \left(\hat{\boldsymbol{Y}}^k(\hat{\boldsymbol{Y}}^k)^*\right)^{\dagger} \in \mathbb{C}^{n_1 n_3 \times \sum_{i=1}^{n_3} \boldsymbol{r}_i^k}$ and $\hat{\boldsymbol{Y}}^{k+1} = \left((\hat{\boldsymbol{X}}^{k+1})^*\hat{\boldsymbol{X}}^{k+1}\right)^{\dagger}(\hat{\boldsymbol{X}}^{k+1})^*\bar{\boldsymbol{C}}^k \in \mathbb{C}^{\sum_{i=1}^{n_3} \boldsymbol{r}_i^k \times n_2 n_3}$. Suppose that $\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{\Sigma}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{V}_{\hat{\boldsymbol{X}}^{k+1}}^*$ and $\boldsymbol{U}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{\Sigma}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*$ are the skinny SVD of $\hat{\boldsymbol{X}}^{k+1}$ and $\hat{\boldsymbol{Y}}^k$, respectively. Then the sequence $\{(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \mathcal{C}^k)\}$ satisfies the following equations:*

$$\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k\|_F^2 = \|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}^*(\bar{\boldsymbol{C}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k)\|_F^2 + \|(\boldsymbol{I}_{n_1 n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}^*)(\bar{\boldsymbol{C}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*\|_F^2 \tag{22}$$

and

$$\|\hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k - \bar{\boldsymbol{C}}^k\|_F^2 - \|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2 = \|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k\|_F^2. \tag{23}$$

Proof. Since $\hat{\boldsymbol{X}}^{k+1} = \boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}^*\hat{\boldsymbol{X}}^{k+1}$ and $\hat{\boldsymbol{Y}}^k = \hat{\boldsymbol{Y}}^k\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*$, we have

$$\begin{aligned}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k &= \bar{\boldsymbol{C}}^k(\hat{\boldsymbol{Y}}^k)^* \left(\hat{\boldsymbol{Y}}^k(\hat{\boldsymbol{Y}}^k)^*\right)^{\dagger}\hat{\boldsymbol{Y}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k \\ &= (\bar{\boldsymbol{C}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*.\end{aligned} \tag{24}$$

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On the other hand, we can obtain the following equation:

$$\begin{aligned}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}=&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}\\ =&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}\hat{\boldsymbol{X}}^{k+1}\left((\hat{\boldsymbol{X}}^{k+1})^*\hat{\boldsymbol{X}}^{k+1}\right)^{\dagger}(\hat{\boldsymbol{X}}^{k+1})^*\bar{\boldsymbol{C}}^k-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}(\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}+\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\\ =&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}\bar{\boldsymbol{C}}^k-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}\left(\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}+(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^k}\right)\\ =&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})(\boldsymbol{I}_{n_2n_3}-\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^k}).\end{aligned}\tag{25}$$

Then the following equation holds:

$$\begin{aligned}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}=&\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}+\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\\ =&(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^k}+\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}).\end{aligned}\tag{26}$$

Note that $\left\langle(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^k},\ \boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\right\rangle=0$, since they are orthogonal to each other. Thus, we can obtain

$$\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2=\|(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^k}\|_F^2+\|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\|_F^2.\tag{27}$$

Therefore, Eq. (22) holds. We can further establish the following equation:

$$\begin{aligned}&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\bar{\boldsymbol{C}}^k\|_F^2\\ =&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}+\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k\|_F^2\\ =&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k\|_F^2+2\left\langle\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k,\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\right\rangle\\ =&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k\|_F^2+2\left\langle\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k,(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^k}\right.\\ &\left.+\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\right\rangle\\ =&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k\|_F^2-2\left(\|(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^k}\|_F^2\right.\\ &\left.+\|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^k-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\|_F^2\right)\\ =&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k\|_F^2-2\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2\\ =&\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k\|_F^2-\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2.\end{aligned}\tag{28}$$

Therefore, Eq. (23) holds. □

Then, we present anther lemma, which will be used later.

Lemma 4. *Suppose that $\boldsymbol{\mathcal{A}}\in\mathbb{R}^{n_1\times n_2\times n_3}$, $\boldsymbol{\mathcal{B}}\in\mathbb{R}^{n_2\times n_4\times n_3}$, $\boldsymbol{\mathcal{F}}\in\mathbb{R}^{n_1\times n_4\times n_3}$ and $\boldsymbol{\mathcal{H}}\in\mathbb{R}^{n_1\times n_4\times n_3}$ are four tensors. If they satisfy the following inequality:*

$$\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{F}}\|_F^2\leq\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{H}}\|_F^2,\tag{29}$$

then we have

$$\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{F}}\|_F^2\leq\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{H}}\|_F^2.\tag{30}$$

Proof. From Lemma 1 in the paper, we know that $\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{F}}$ and $\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{F}}$ are equivalent to each other. $\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{H}}$ and $\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{H}}$ are also equivalent. Thus, we can obtain $\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{F}}\|_F^2=\frac{1}{n_3}\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{F}}\|_F^2$ and $\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{H}}\|_F^2=\frac{1}{n_3}\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{H}}\|_F^2$. Thus, if inequality (29) holds, then inequality (30) holds. □

Now, we prove Theorem 2.

Proof. Assume that $f(\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}},\boldsymbol{\mathcal{C}})=\frac{1}{2n_3}\|\hat{\boldsymbol{X}}\hat{\boldsymbol{Y}}-\bar{\boldsymbol{C}}\|$ is the objective function. From Lemma 3, the following equation holds.

$$\begin{aligned}f(\hat{\boldsymbol{X}}^k,\hat{\boldsymbol{Y}}^k,\boldsymbol{\mathcal{C}}^k)-f(\hat{\boldsymbol{X}}^{k+1},\hat{\boldsymbol{Y}}^{k+1},\boldsymbol{\mathcal{C}}^k)=&\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^k\|_F^2-\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\bar{\boldsymbol{C}}^k\|_F^2\\ =&\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2.\end{aligned}\tag{31}$$

On the other hand, we note that $\boldsymbol{\mathcal{C}}^{k+1}$ is the optimal solution to problem (5) in the paper:

$$\boldsymbol{\mathcal{C}}^{k+1} = \underset{P_\Omega(\boldsymbol{\mathcal{C}}-\boldsymbol{\mathcal{M}})=\mathbf{0}}{\operatorname{argmin}} \|\boldsymbol{\mathcal{X}}^{k+1} * \boldsymbol{\mathcal{Y}}^{k+1} - \boldsymbol{\mathcal{C}}\|_F^2. \tag{32}$$

At the same time, we note that $P_\Omega(\boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{M}}) = 0$, *i.e.*, $\boldsymbol{\mathcal{C}}^k$ is a feasible solution to problem (32). So the following inequality holds.

$$\|\boldsymbol{\mathcal{X}}^{k+1} * \boldsymbol{\mathcal{Y}}^{k+1} - \boldsymbol{\mathcal{C}}^{k+1}\|_F^2 \leq \|\boldsymbol{\mathcal{X}}^{k+1} * \boldsymbol{\mathcal{Y}}^{k+1} - \boldsymbol{\mathcal{C}}^{k}\|_F^2, \tag{33}$$

From Lemma 4, we can obtain

$$\|\bar{\boldsymbol{X}}^{k+1}\bar{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2 \leq \|\bar{\boldsymbol{X}}^{k+1}\bar{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k}\|_F^2. \tag{34}$$

Since $\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} = \bar{\boldsymbol{X}}^{k+1}\bar{\boldsymbol{Y}}^{k+1}$, we have

$$\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2 \leq \|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k}\|_F^2. \tag{35}$$

Then, it follows that

$$\begin{aligned}
&f(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \boldsymbol{\mathcal{C}}^k) - f(\hat{\boldsymbol{X}}^{k+1}, \hat{\boldsymbol{Y}}^{k+1}, \boldsymbol{\mathcal{C}}^{k+1})\\
=&\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k - \bar{\boldsymbol{C}}^k\|_F^2 - \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2\\
=&\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k - \bar{\boldsymbol{C}}^k\|_F^2 - \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2 + \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2 - \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2\\
\geq&\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k\|_F^2.
\end{aligned} \tag{36}$$

Summing all the inequality (36) for all k, we obtain

$$f(\hat{\boldsymbol{X}}^1, \hat{\boldsymbol{Y}}^1, \boldsymbol{\mathcal{C}}^1) - f(\hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \boldsymbol{\mathcal{C}}^n) = \frac{1}{2n_3}\sum_{i=1}^{n}\|\hat{\boldsymbol{X}}^{i+1}\hat{\boldsymbol{Y}}^{i+1} - \hat{\boldsymbol{X}}^i\hat{\boldsymbol{Y}}^i\|_F^2 < +\infty. \tag{37}$$

Thus, we can obtain the following equation:

$$\lim_{n\to+\infty}\|\hat{\boldsymbol{X}}^{n+1}\hat{\boldsymbol{Y}}^{n+1} - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n\|_F^2 = 0, \tag{38}$$

Assume that $\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{\Sigma}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{V}^*_{\hat{\boldsymbol{X}}^{n+1}}$ and $\boldsymbol{U}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{\Sigma}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}$ are the skinny SVD of $\hat{\boldsymbol{X}}^{n+1}$ and $\hat{\boldsymbol{Y}}^n$, respectively. From Lemma 3, we can further obtain

$$\lim_{n\to+\infty}\|(\boldsymbol{I}_{n_1n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}})(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}\|_F^2 + \|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\|_F^2 = 0, \tag{39}$$

So, the following two equations hold:

$$\lim_{n\to+\infty}\|(\boldsymbol{I}_{n_1n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}})(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}\|_F^2 = 0 \tag{40}$$

and

$$\lim_{n\to+\infty}\|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\|_F^2 = 0. \tag{41}$$

We can further establish the following equations:

$$\lim_{n\to+\infty}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n) = \mathbf{0}. \tag{42}$$

Since $\hat{\boldsymbol{Y}}^n$ is bounded, $\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}$ is bounded. Thus, we can establish the following equation:

$$\lim_{n\to+\infty}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n} = \mathbf{0}. \tag{43}$$

So we can obtain

$$\mathbf{0} = \lim_{n\to+\infty}(\boldsymbol{I}_{n_1n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}})(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n} = \lim_{n\to+\infty}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}. \tag{44}$$

Since $(\hat{\boldsymbol{Y}}^n)^* = \boldsymbol{V}_{\hat{\boldsymbol{Y}}_n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}_n}(\hat{\boldsymbol{Y}}_n)^*$, $(\hat{\boldsymbol{X}}^{n+1})^* = (\hat{\boldsymbol{X}}^{n+1})^*\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}$, and $\hat{\boldsymbol{Y}}^n$, $\hat{\boldsymbol{X}}^{n+1}$ are bounded, we have

$$\mathbf{0} = \lim_{n\to+\infty}(\hat{\boldsymbol{X}}^{n+1})^*\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n) = \lim_{n\to+\infty}(\hat{\boldsymbol{X}}^{n+1})^*(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n) \tag{45}$$

and

$$\mathbf{0} = \lim_{n \to +\infty} (\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n \hat{\boldsymbol{Y}}^n) \boldsymbol{V}_{\hat{\boldsymbol{Y}}^n} \boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}^* (\hat{\boldsymbol{Y}}^n)^* = \lim_{n \to +\infty} (\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n \hat{\boldsymbol{Y}}^n)(\hat{\boldsymbol{Y}}^n)^*. \tag{46}$$

Since the sequence $\{(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \boldsymbol{\mathcal{C}}^k)\}$ generated by our algorithm is bounded, there is a subsequence $\{(\hat{\boldsymbol{X}}^{k_j}, \hat{\boldsymbol{Y}}^{k_j}, \boldsymbol{\mathcal{C}}^{k_j})\}$ that converges to a point $(\hat{\boldsymbol{X}}_\star, \hat{\boldsymbol{Y}}_\star, \boldsymbol{\mathcal{C}}_\star)$. Therefore, the following two equations hold:

$$(\bar{\boldsymbol{C}}_\star - \hat{\boldsymbol{X}}_\star \hat{\boldsymbol{Y}}_\star)(\hat{\boldsymbol{Y}}_\star)^* = \mathbf{0}, \tag{47}$$

$$(\hat{\boldsymbol{X}}_\star)_*(\bar{\boldsymbol{C}}_\star - \hat{\boldsymbol{X}}_\star \hat{\boldsymbol{Y}}_\star) = \mathbf{0}. \tag{48}$$

On the other hand, we update $\boldsymbol{\mathcal{C}}^{k+1} = \boldsymbol{\mathcal{X}}^k * \boldsymbol{\mathcal{Y}}^k + P_\Omega(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{X}}^k * \boldsymbol{\mathcal{Y}}^k)$ at each iteration. Thus, $\boldsymbol{\mathcal{C}}_\star$ always satisfies the following two equations.

$$\begin{aligned} &P_{\Omega^c}(\boldsymbol{\mathcal{C}}_\star - \boldsymbol{\mathcal{X}}_\star * \boldsymbol{\mathcal{Y}}_\star) = \mathbf{0}, \\ &P_\Omega(\boldsymbol{\mathcal{C}}_\star - \boldsymbol{\mathcal{M}}) = \mathbf{0}. \end{aligned} \tag{49}$$

And we can always find $\boldsymbol{\mathcal{Q}}_\star$ that meets the following equations.

$$P_\Omega(\boldsymbol{\mathcal{C}}_\star - \boldsymbol{\mathcal{X}}_\star * \boldsymbol{\mathcal{Y}}_\star) + \boldsymbol{\mathcal{Q}}_\star = \mathbf{0}. \tag{50}$$

So $(\hat{\boldsymbol{X}}_\star, \hat{\boldsymbol{Y}}_\star, \boldsymbol{\mathcal{C}}_\star)$ is a KKT point of problem (13) in the paper. $\square$